Topology-II
M. Math - I
Mid-Semestral Exam
2011-2012

Time: 3hrs
Max score: 100

Answer all questions.

(1) (i)State Siefert-van Kampen theorem.
(ii) Show that fundamental group of n circles joined at a point is the free group F_n on n generators.
(iii) Let $L_1, \cdots, L_m$ be m lines passing through the origin in R^3 and $X = \bigcup_{j=1}^{m} L_j$. Show that the sphere S^2 with $2m$ points removed is a deformation retract of $R^3 - X$. Hence, show that $\pi_1(R^3 - X) \cong F_{2m-1}$.

(8+5+12)

(2) (i) Define **equivalence of covering spaces**.
(ii) Show that the covering maps $p : E \longrightarrow B$ and $p' : E' \longrightarrow B$ with $p(e_0) = p'(e_0') = b_0$ are equivalent if and only if the subgroups

$$H_0 = p_*(\pi_1(E, e_0)) \text{ and } H_0' = p_*'(\pi_1(E', e_0'))$$

of $\pi_1(B, b_0)$ are conjugate.
(iii) Let $T = S^1 \times S^1$. There is an isomorphism of $\pi_1(T, b_0 \times b_0)$ with $\mathbb{Z} \times \mathbb{Z}$ induced by projections of T onto its two factors. Find a covering space of T corresponding to the subgroup of $\mathbb{Z} \times \mathbb{Z}$ generated by $m \times 0$ and $0 \times n$, where m and n are positive integers.

(5+10+10)

(3) Assume $p : E \longrightarrow B$ is a covering map with $p(e_0) = b_0$.
(i) Define the **group of covering transformations**.
(ii) If $H_0 = p_*(\pi_1(E, e_0))$, show that the group of covering transformations is isomorphic to $N(H_0)/H_0$, where $N(H_0)$ is the normalizer of H_0 in $\pi_1(B, b_0)$.
(iii) Show that the projection map $q : \mathbb{S}^n \longrightarrow \mathbb{R}P^n$ is a covering map. Also show that the group of covering transformations of this covering map is $\mathbb{Z}_2$.

(3+12+10)

(4) (i) Let G be a group of homeomorphisms of X. Define **properly discontinuous action** of G on X.
(ii) Let X be path connected and locally path connected and G be a group of homeomorphisms of X. Show that the quotient map $\pi : X \longrightarrow X/G$ is a covering map if and only if the action of G is properly discontinuous. Also show that in this case the covering map π is regular and G is its group of covering transformations.

(3+22)